UDC 62-50

ON THE RELATION BETWEEN THE SOLVABILITY CONDITIONS OF DIFFERENTIAL GAMES

PMM Vol. 40, № 2, 1976, pp. 213-221
V. D. BATUKHTIN
(Sverdlovsk)
(Received June 9, 1975)

We examine position encounter-evasion problems for inherently linear controlled systems in which the players' controls do not separate additively. We propose construction permitting effective solutions to these problems in the classes of pure and mixed strategies and of counterstrategies [1, 2]. We investigate the relation between the proposed and the program constructions for solving such problems. The constructions used in this paper go back to the direct method [3—6]. In contrast to [3—6] the approach described here includes the solving of position problems.

1. Let the motion of a conflict-controlled system be described by the inherently linear differential equation

$$dx/dt = A(t)x + f(t, u, v), \quad x(t_0) = x_0; \quad u \in P \subset R^p \qquad (1.1)$$
$$v \in Q \subset R^q$$

Here x is an n-dimensional phase vector, $A(t)$ is an $(n \times n)$-dimensional continuous matrix function, $f(t, u, v)$ is a vector function continuous in all arguments, P and Q are compacta. A convex compactum M in space R^m and the function $r(x, m) = \|x - m\|$ are specified, ($\|p\|$ is the Euclidean norm of vector p, $m \in M$). The game's outcome is determined by the functional

$$\varphi(x[\cdot]) = \min_{t \in [t_0, \vartheta]} \omega(x[t]) \qquad (\omega(x) = \min_{m \in M} r(x, m)) \qquad (1.2)$$

where ϑ is some finite instant. The first player strives to minimize functional (1.2) and the second player, to maximize it. We assume that the players' pure strategies U_1 and V_1, mixed strategies U_2 and V_2, counterstrategies U_3 and V_3 and the limit transition generated by them from the corresponding Euler polygonal lines of the motion $x_{U_i}[t]$ ($x_{V_i}[t]$), $i = 1,2,3$, are defined in the same way as in [1, 2].

Problem 1. For a fixed position $\{t_0, x_0\}$ find the optimal minimax strategy $U_1^\circ \div u^\circ(t, x)$ or the optimal minimax mixed strategy $U_2^\circ \div \mu(du / t, x)$ or the optimal minimax counterstrategy $U_3^\circ \div u^\circ(t, x, v)$ which satisfies the condition

$$\sup_{x[\cdot]} \varphi(x[\cdot; t_0 x_0 U_i^\circ]) = \min_{U_i} \sup_{x[\cdot]} \varphi(x[\cdot; t_0, x_0, U_i]), \qquad i = 1, 2, 3 \qquad (1.3)$$

on any motion $x_{U_i^\circ}[t] = x[t; t_0, x_0, U_i^\circ]$.

Problem 2. For a fixed position $\{t_0, x_0\}$ find the optimal maximin strategy $V_1^\circ \div v^\circ(t, x)$ or the optimal maximin mixed strategy $V_2^\circ \div \nu^\circ(dv / t, x)$ or the optimal maximin counterstrategy $V_3^\circ \div v^\circ(t, x, u)$ which satisfies condition

$$\inf_{x[\cdot]} \varphi(x[\cdot; t_0, x_0, V_i^\circ]) = \max_{V_i} \inf_{x[\cdot]} \varphi(x[\cdot; t_0, x_0, V_i]), \qquad i = 1, 2, 3 \qquad (1.4)$$

on any motion $x_{V_i^\circ}[t] = x[t;\ t_0,\ x_0,\ V_i^\circ]$.

2. Let us consider the connection between the solvability conditions for Problems 1 and 2 in the form of extremal aiming and using an a priori stable bridge [1, 2]. For definiteness we present the reasonings here for the minimax formulation, i.e. for the pair strategy U_1-counterstrategy V_3. The corresponding statements, as applied to a differential game in mixed strategies and in the class counterstrategy U_3 — strategy V_1 are verified similarly; we present them below without proof.

We introduce an auxiliary program construction. Namely, by the symbol V_u we denote the counterstrategy-program [2] which associates with every pair $\{t,\ u\}$ a set $V(t,\ u) \in Q$ upper-semicontinuous with respect to inclusion relative to $\{t,\ u\}$. We determine a program motion $x(t) = x(t;\ t_*,\ x_*,\ V_u)$ $(t_* \in [t_0,\ \vartheta])$ as a solution of the contingent differential equation

$$x^\cdot(t) \in A(t)\, x(t) + F(t,\ V_u); \tag{2.1}$$

$$F(t,\ V_u) = \mathrm{co}\,\{f : f = f(t,\ u,\ v),\ v \in V(t,\ u),\ u \in P\}$$

where co $\{f\}$ is the closed convex hull of the set $\{f\}$ of vectors f. Further, we denote

$$\varepsilon_1^\circ(t_*, x_*) = \varepsilon_1^\circ(t_*, x_*; \tau_\circ) = \min_{\tau \in [t_*, \vartheta]} \varepsilon_1^\circ(t_*, x_*; \tau) \tag{2.2}$$

$$\varepsilon_1^\circ(t_*, x_*; \tau) = \max_{V_u}\ \min_{x(\cdot)} \omega(x(\tau; t_*, x_*, V_u)) \tag{2.3}$$

It can be verified that the quantity $\varepsilon_1^\circ(t_*,\ x_*;\ \tau)$ in (2.3) can be represented by the equality

$$\varepsilon_1^\circ(t_*, x_*; \tau) = \max_{V_u}\ \max_{\|l\|=1} \Big[l'\{X(\tau, t_*)\, x_*\}_m + \int_{t_*}^{\tau} \Big(\min_{f \in F(t, V_u)} l'\{X(\tau, t) f\}_m \Big) dt + \rho_M(l) \Big] \tag{2.4}$$

if the right-hand side of this equality is positive; otherwise $\varepsilon_1^\circ(t_*,\ x_*;\ \tau) = 0$. Here l is an m-dimensional vector, $X(t,\ \tau)$ is the transition matrix for the solutions of the equation $x^\cdot = A(t)\, x$, $X(\tau,\ \tau) = E$, $\rho_M(l) = \min l'm$ for $(-m) \in M$, the prime denotes transposition. We note that the optimal program $V_u^\circ = V^\circ(t,\ u)$, implying the maximum in (2.4), exists, while the expression (2.4) for $\varepsilon_1^\circ(t_*,\ x_*;\ \tau)$ can be written as

$$\varepsilon_1^\circ(t_*, x_*; \tau) = \max_{\|l\|=1} \Big[l'\{X(\tau, t_*)\, x_*\}_m + \int_{t_*}^{\tau} \min_{u \in P}\ \max_{v \in Q} l'\{X(\tau, t)\, f(t, u, v)\}_m\, dt + \rho_M(l) \Big] \tag{2.5}$$

We say that the game (1.1), (1.2) is regular if the following conditions are fulfilled.

Condition A1. For every position $\{t_*,\ x_*\}$ $(t_0 \leqslant t_* \leqslant \vartheta,\ t_* \neq \tau_0,\ \varepsilon_1^\circ(t_*,\ x_*) \in (0, \beta),\ \beta = \mathrm{const})$, for any choice of function $v(u)$ we can find at least one instant $\tau_0 \in [t_*,\ \vartheta]$ and one vector $f_* \in \mathrm{co}\,\{f : f = f(t,\ u,\ v(u)),\ u \in P\}$ such that the inequality

$$l_*^{\circ\prime} X(\tau_0, t_*) f_* \leqslant \min_{u \in P}\ \max_{v \in Q} l_*^{\circ\prime} X(\tau_0, t_*) f(t_*, u, v),\ l_*^{\circ\prime} = \{l^{\circ\prime}, 0, 0, \ldots, 0\}$$

is valid for any of the maximizing vectors l° in (2.4), corresponding to the instant τ_0.

Condition B1. For every position $\{t_*, x_*\}$ $(t_0 \leqslant t_* \leqslant \vartheta,\ \varepsilon_1^\circ(t_*, x_*) \in (0, \beta)$, for any choice of vector $u \in P$ we can find at least one vector $f^* \in \mathrm{co}\ \{f : f = f(t, u, v),\ v \in Q\}$ such that the inequality

$$l_*^{\circ\prime} X(\tau_0, t_*) f^* = \max_{v \in Q} l_*^{\circ\prime} X(\tau_0, t_*) f(t_*, u, v)$$

is valid for any of the maximizing vectors l° in (2.4), corresponding to any of the minimizing instants τ_0 in (2.2).

We note that Conditions A 1 and B 1 correspond, respectively, to Conditions A, B and C of [7], but now in application to the inherently linear system (1.1) being analyzed. The validity of the next assertion is proved by reasonings similar to those in [7].

Theorem 2.1. Let the regularity Conditions A 1 and B.1 be fulfilled. Then the strategy $U_1^\circ \div u^\circ(t, x)$, extremal [1, 2] to the set $W_{1,\varepsilon} = \{\{t,x\} : \varepsilon_1^\circ(t, x) \leqslant \varepsilon_1^\circ(t_0, x_0)\}$, solves Problem 1 in the minimax case, guaranteeing an encounter of every motion $x[t;\ t_0,\ x_0,\ U_1^\circ]$ of system (1.1) with set M when $\varepsilon_1^\circ(t_0, x_0) = 0$. The counterstrategy $V_3^\circ \div v^\circ(t, x, u)$, extremal to the set $W_1^\varepsilon = \{\{t, x\}: \varepsilon_1^\circ(t, x) > \varepsilon_1^\circ(t_0, x_0)\}$, solves Problem 2 in the minimax case.

We now consider the connection between the regularity Conditions A 1 and B 1 and the following two conditions.

Condition C1. The function

$$\varkappa_1(t, \tau, l) = -\min_{u \in P} \max_{v \in Q} l' \{X(\tau, t) f(t, u, v)\}_m$$

is convex in l for all $t \in [t_0, \tau]$ and for all $\tau \in [t_0, \vartheta]$.

Condition D1. For every vector $u \in P$ we can find a vector $f_{t,u}^* \in \mathrm{co}\ \{f : f = f(t, u, v),\ u \in P\}$ such that we have

$$l' \{X(\tau, t) f_{t,u}^*\}_m = \max_{v \in Q} l' \{X(\tau, t) f(t, u, v)\}_m$$

for all τ and t $(t_0 \leqslant t \leqslant \tau \leqslant \vartheta)$ and for all m-dimensional vectors l .

By the symbol $H_1(\tau, t)$ we denote the set defined by the relation

$$H_1(\tau, t) = \bigcap_{v(\cdot)} \mathrm{co}\,\{\{X(\tau, t) f(t, u, v(u))\}_m : u \in P\} \tag{2.6}$$

where the function $v(u) \in Q$ within the braces is fixed, while the vector u ranges over the whole set P; the intersection is taken over all possible functions $v(u) \in Q$. It can be verified that an m-dimensional vector h belongs to set $H_1(\tau, t)$ if and only if it satisfies the condition

$$\max_{\|l\|=1} [\min_{u \in P} \max_{v \in Q} l' \{X(\tau, t) f(t, u, v)\}_m - l'h] \leqslant 0 \tag{2.7}$$

When Condition C1 is satisfied, a vector h, for which condition (2.7) is satisfied exists, as is easy to show, and consequently, the sets $H_1(\tau, t)$ are nonempty in this case.

We introduce, further, the quantity $\varepsilon_1^*(t_*, x_*)$ defined by the equality

$$\varepsilon_1^*(t_*, x_*) = \min_{\tau \in [t_*, \vartheta]} \min_{x(\cdot)} \min_{m \in M} r(x(\tau; t_*, x_*), m) \tag{2.8}$$

where $x(t;\ t_*,\ x_*)$ is a motion, being a solution of the contingent Eq. (2.9) under constraint (2.10)

$$x^\cdot(t) = A(t)\,x(t) + g(t) \tag{2.9}$$

$$\{X(\tau,\ t)\ g(t)\}_m \in H_1(\tau,\ t) \tag{2.10}$$

We note that from the construction of the sets $H_1(\tau,\ t)$ it follows that the sets

$$W_t^{g(\cdot)} = \{x\colon\ \{X(\tau,\ t)\ x_{g(\cdot)}(t)\}_m = \{X(\tau,\ t)\ x\}_m\}$$

where $x_{g(\cdot)}(t)$ is a solution of Eq. (2.9) under "control" $g(t)$, satisfying (2.10), form a minimax u-stable bridge. Therefore, if for the initial position $\{t_0,\ x_0\}$ we can find a solution $x_{g(\cdot)}(t)$ of Eq. (2.9) satisfying the condition $x_{g(\cdot)}(\tau) \in M$ for some $\tau \in [t_0,\ \vartheta]$, then the strategy $U^{(e)} \div u^{(e)}(t,\ x)$ extremal [1, 2] to this bridge $W_t^{g(\cdot)}$ ensures the encounter of system (1.1), (1.2) with set M by this instant. The following assertion is valid.

Lemma 2.1. Let Condition C1 be satisfied. Then the equality

$$\varepsilon_1^\circ(t_*,\ x_*) = \varepsilon_1^*(t_*,\ x_*) \tag{2.11}$$

is valid.

The lemma is proved on the following plan. Let $\tau = \tau^\circ$ be the minimizing instant defined by relation (2.8) and let $x^*(t;\ t_*,\ x_*) = x_{g^*(\cdot)}(t;\ t_*,\ x_*)\ (t_* \leqslant t \leqslant \tau^\circ)$ be the minimizing motion from (2.8), corresponding to this instant. From the fact that the bridge $W_t^{g^*(\cdot)}$ is minimax u-stable, we deduce the existence of a program motion $x(t;\ t_*,\ x_*,\ V_u)$, for each program V_u, such that

$$\varepsilon_1^*(t_*,\ x_*) = \omega(x(\tau^\circ;\ t_*,\ x_*)) = \omega(x(\tau^\circ;\ t_*,\ x_*,\ V_u))$$

Hence follows the validity of the inequality

$$\varepsilon_1^\circ(t_*,\ x_*) \leqslant \varepsilon_1^*(t_*,\ x_*) \tag{2.12}$$

Let us show the validity of the relation inverse to inequality (2.12)

$$\varepsilon_1^\circ(t_*,\ x_*) \geqslant \varepsilon_1^*(t_*,\ x_*) \tag{2.13}$$

The required assertion will then follow from (2.12) and (2.13).

In fact, expression (2.8) for $\varepsilon_1^*(t_*,\ x_*)$ can be written as

$$\varepsilon_1^*(t_*,x_*) = \max_{\|l\|=1}\left[l'\{X(\tau^\circ,t_*)\,x_*\}_m + \int_{t_*}^{\tau^\circ} \min_{h\in H_1(\tau^\circ,t)} l'h(t)\,dt + \rho_M(l)\right] \tag{2.14}$$

By l^* and $h^*(t)$ we denote an m-dimensional vector and a vector function implying, respectively, the maximum and the minimum in (2.14). We take a vector function $u^*(t)$ which satisfies the minimax condition

$$\max_{v\in Q} l'\{X(\tau^\circ,t)\,f(t,u^*(t),v)\}_m = \min_{u\in P}\max_{v\in Q} l^{*\prime}\{X(\tau^\circ,t)\,f(t,u,v)\}_m \tag{2.15}$$

for almost all $t \in [t_*,\ \tau^\circ]$ and for each $t \in [t_*,\ \tau^\circ]$ we find the vector $f_{t,\,u^*(t)}$ for which the equality

$$\{X(\tau^\circ,t)\,f_{t,u^*(t)}\}_m = h(t) \tag{2.16}$$

is satisfied. The existence of vector $h(t) \in H_1(\tau^\circ,\ t)$ in (2.16) follows from the fact that under Condition C1 the intersection of the set $H_1(\tau^\circ,\ t)$ with any of the sets $F_v(t,\ u) = \mathrm{co}\ \{\{X(\tau^\circ,\ t)\ f(t,\ u,\ v)\}_m\colon v \in Q\}$ is nonempty for every fixed $u \in P$. From (2.14)—(2.16) we obtain

$$l^{*\prime}h(t) \leqslant l^{*\prime}\{X(\tau^\circ,t)\,f_{t,u^*(t)}\}_m \leqslant \min_{u\in P}\max_{v\in Q} l^{*\prime}\{X(\tau^\circ,t)\,f(t,u,v)\}_m$$

whence the required equality (2.13) is derived with due regard to (2.5).

The following statement is valid:

Theorem 2.2. Let Conditions C1 and D1 be fulfilled. From Conditions C1 and D1 follow the regularity Conditions A1 and B1 for game (1.1),(1.2) and the equality $\varepsilon_1^\circ(t_0, x_0) = \varepsilon_1^*(t_0, x_0)$.

In fact, from the convexity of the function $\varkappa_1(\tau, t, l)$ it follows that the maximizing vector l° in (2.5) is unique. But then Condition A1 is automatically satisfied. Condition B1 follows directly from Condition D1, and the vector f^* dealt with in D1 satisfies the corresponding equality for all values of vector l and not only for the vectors l°. The equality $\varepsilon_1^\circ(t_0, x_0) = \varepsilon_1^*(t_0, x_0)$ has been proved in Lemma 2.1.

Thus, when Conditions C1 and D1 are satisfied, the problem of encounter by the instant τ° (Problem 1) can be solved either by means of the sets $H_1(\tau^\circ, t)$ of (2.6) or by means of a minimax u-stable bridge $W_{1,\varepsilon}$ of program absorption. Furthermore, under Condition C1 the function $\varepsilon_1^\circ(t, x; \tau)$ is differentiable [8] in the region $\varepsilon_1^\circ(t, x) \in (0, \beta)$ for each fixed τ; therefore, in the case being examined the encounter problem is effectively solvable by the extremal aiming strategy [8]. When Condition D1 is satisfied, the evasion problem (Problem 2) can therefore be solved by using the v-stable program bridge W_1^ε. Moreover, it turns out that when Conditions C1 and D1 are satisfied, the evasion problem is effectively solvable by the counterstrategy of generalized extremal aiming. We define this counterstrategy in the following manner.

We introduce the function

$$\lambda_1(t, x) = \int_t^\vartheta \frac{1}{\varepsilon_1^\circ(t, x; \tau)}\, d\tau$$

By G_1 we denote the region of positions $\{t, x\}$ for which $\varepsilon_1^\circ(t, x) \in (0, \beta)$. Since for each τ the function $\varepsilon_1^\circ(t, x; \tau)$ is differentiable in region G_1, the function $\lambda_1(t, x)$ is differentiable in this region. If $\{t, x\} \notin G_1$, the counterstrategy $V^{(e)}(t, x, u)$ of generalized extremal aiming is identified with any function $v(t, x, u) \in Q$ Borel in u; if, however, $\{t, x\} \in G_1$, then $V^{(e)}(t, x, u)$ is identified with any function $v^{(e)} = v^{(e)}(t, x, u) \in Q$ Borel in u, for which the maximum condition

$$\left[-\frac{\partial \lambda_1(t, x)}{\partial x}\right]' f(t, u, v^{(e)}) = \max_{v \in Q} \left[-\frac{\partial \lambda_1(t, x)}{\partial x}\right]' f(t, u, v)$$

is satisfied, where $[\partial\lambda_1(t, x) / \partial x]$ is the matrix of partial derivatives $\partial\lambda_1(t, x) / \partial x_i$, $i = 1, 2, \ldots, n$. We write the contingent equation (2.17)

$$x^{\boldsymbol{\cdot}} \in A(t)x + \mathrm{co}\,\{f : f = f(t, u, v(u)),\ v(u) \in V^{(e)}(t, x, u), u \in P\}$$

The following assertion is valid.

Theorem 2.3. Let Conditions C1 and D1 be satisfied. Then the counterstrategy $V^{(e)}(t, x, u)$ of generalized extremal aiming ensures the evasion of all motions $x[t] = x[t; t_0, x_0, V^{(e)}]$ in (2.17) from the set M on the interval $[t_0, \vartheta]$ when $\varepsilon_1^\circ(t_0, x_0) \in (0, \beta)$.

Directly from Theorems 2.2 and 2.3 follows

Theorem 2.4. Let Conditions C1 and D1 be satisfied and let $\varepsilon_1^\circ(t_0, x_0) \in (0, \beta)$. Then $\varepsilon_1^\circ(t_0, x_0) = \varepsilon_1^*(t_0, x_0)$ is the value of the encounter-evasion game.

3. Here we state without proof assertions analogous to those presented in Sect. 2 in connection with the solution of Problems 1 and 2 in the class of mixed strategies $\{U_2, V_2\}$ and for the pair $\{U_3, V_1\}$. We denote

$$, \varepsilon_i^\circ(t_*, x_*; \tau) = \max_{\|l\|=1} [l'\{X(\tau, t_*) x_*\}_m + \rho_i(t_*, \tau; l) + \rho_M(l)] \tag{3.1}$$

$$\varepsilon_i^\circ(t_*, x_*) = \varepsilon_i^\circ(t_*, x_*; \tau_0) = \min_{\tau\in[t_*,\vartheta]} \varepsilon_i^\circ(t_*, x_*; \tau), \quad i = 2, 3 \tag{3.2}$$

$$\rho_2(t_*, x_*, \tau) = \int_{t_*}^{\tau} \Big[\min_{\mu} \max_{\nu} \int_P \int_Q l'\{X(\tau, t) f(t, u, v)\}_m \mu(du) \nu(dv)\Big] dt$$

$$\rho_3(t_*, x_*; \tau) = \int_{t_*}^{\tau} \max_{v\in Q} \min_{u\in P} l'\{X(\tau, t) f(t, u, v)\}_m dt$$

where $\mu(du)$ and $\nu(dv)$ are probability measures on P and Q, respectively. We say that in the cases being considered the game (1.1),(1.2) is regular if the following conditions are satisfied, respectively.

Condition A2 (A3). For every position $\{t_*, x_*\}$ $(t_0 \leqslant t_* \leqslant \vartheta,\ t_* \neq \tau_0$, $\varepsilon_i^\circ(t_*, x_*) \in (0, \beta),\ i = 2, 3)$, for any choice of probability measure $\nu_*(dv)$ (of vector $v \in Q$) we can find at least one instant $\tau_0 \in [t_*, \vartheta]$ and one probability measure $\mu_*(du)$ (vector $f_* \in \mathrm{co}\,\{f : f = f(t, u(v), v), u(v) \in P\}$) such that the relation

$$\int_P\int_Q l_*^{\circ\prime} X(\tau_0, t_*) f(t_*, u, v) \mu_*(du) \nu_*(dv) \leqslant$$
$$\min_{\mu} \max_{\nu} \int_P \int_Q l_*^{\circ\prime} X(\tau_0, t_*) f(t_*, u, v) \mu(du) \nu(dv)$$
$$(l_*^{\circ\prime} X(\tau_0, t_*) f_* \leqslant \max_{v\in Q} \min_{u\in P} l_*^{\circ\prime} X(\tau_0, t_*) f(t_*, u, v))$$

is valid for any of the minimizing vectors l° in (3.1), correponding to instant τ_0

Condition B2 (B3). For every position $\{t_*, x_*\}$ $(t_0 \leqslant t_* \leqslant \vartheta,\ \varepsilon_i^\circ(t_*, x_*) \in (0, \beta),\ i = 2, 3)$, for any choice of probability measure $\mu^*(du)$ (of function $u(v)$) we can find at least one probability measure $\nu^*(dv)$ (vector $f^* \in \mathrm{co}\,\{f : f = f(t, u, v), u = u(v), v \in Q\}$) such that the relation

$$\int_P\int_Q l_*^{\circ\prime} X(\tau_0, t_*) f(t_*, u, v) \mu^*(du) \nu^*(dv) \geqslant$$
$$\max_{\nu} \min_{\mu} \int_P \int_Q l_*^{\circ\prime} X(\tau_0, t_*) f(t_*, u, v) \mu(du) \nu(dv)$$
$$(l_*^{\circ\prime} X(\tau_0, t_*) f^* \geqslant \max_{v\in Q} \min_{u\in P} l_*^{\circ\prime} X(\tau_0, t_*) f(t_*, u, v))$$

is valid for any of the maximizing vectors l° in (3.1), corresponding to any of the minimizing instants τ_0 in (3.2).

An assertion, similar to Theorem 2.1, on the solvability of Problems 1 and 2 in the class of mixed strategies $\{U_2, V_2\}$ (in the class counterstrategy U_3 — strategy V_1) is valid when the regularity Conditions A2 and B2 (A3 and B3) are satisfied.

For the cases being examined let us now formulate the solvability conditions for Problems 1 and 2 in a form similar to Conditions C1 and D1. Here these conditions take the form:

Condition C2 (C3). The function

$$\varkappa_2(t,\tau,l) = -\min_{\mu}\max_{\nu}\int_P\int_Q l'\{X(\tau,t)f(t,u,v)\}_m\,\mu(du)\,\nu(dv)$$

$$(\varkappa_3(t,\tau,l) = -\max_{v\in Q}\min_{u\in P} l'\{X(\tau,t)f(t,u,v)\}_m)$$

is convex in l for all $t\in[t_0,\tau]$ and for all $\tau\in[t_0,\vartheta]$.

Condition D2(D3). For every measure μ (du) (function $u(v)\in P$) we can find a measure ν (dv) (vector $f^*\in \mathrm{co}\,\{f: f=f(t,\ u(v),\ v),\ v\in Q\}$) such that we have

$$\int_P\int_Q l'\{X(\tau,t)f(t,u,v)\}_m\,\mu(du)\,\nu(dv)\geqslant$$

$$\min_{\mu}\max_{\nu}\int_P\int_Q l'\{X(\tau,t)f(t,u,v)\}_m\,\mu(du)\,\nu(dv)$$

$$(l'\{X(\tau,t)f^*\}_m\geqslant\max_{v\in Q}\min_{u\in P} l'\{X(\tau,t)f(t,u,v)\}_m)$$

for all τ and t $(t_0\leqslant t\leqslant\tau\leqslant\vartheta)$ and for all m-dimensional vectors l.

Introducing the sets defined by the relations

$$H_2(\tau,t)=\bigcap_{\nu(dv)}\Big[\bigcup_{\mu(du)}\int_P\int_Q\{X(\tau,t)f(t,u,v)\}_m\,\mu(du)\,\nu\,(dv)\Big]$$

$$H_3(\tau,t)=\bigcap_{v\in Q}\mathrm{co}\,\{\{X(\tau,t)f(t,u,v)\}_m: u\in P\}$$

we can verify the validity of the next statement by using arguments similar to those in Sect. 2.

Theorem 3.1. Let conditions C2 and D2 (C3 and D3) be satisfied. Then the regularity Conditions A2 and B2 (A3 and B3) for game (1.1), (1.2) are satisfied automatically and $\varepsilon_2^\circ(t_0,x_0)=\varepsilon_2^*(t_0,x_0)$ $(\varepsilon_3^\circ(t_0,x_0)=\varepsilon_3^*(t_0,x_0))$. The mixed strategy $V_2^{(e)}(t,\ x)$ of generalized extremal aiming when $\varepsilon_2^\circ(t_0,x_0)\in(0,\ \beta)$ (the generalized strategy $V_1^{(e)}(t,\ x)$ when $\varepsilon_3^\circ(t_0,x_0)\in(0,\ \beta)$) ensures the evasion of all motions $x[t;t_0,x_0,V_2^{(e)}]$ $(x[t;t_0,x_0,V_1^{(e)}])$ from set M on $[t_0,\ \vartheta]$. The quantity $\varepsilon_2^\circ(t_0,\ x_0)=\varepsilon_2^*(t_0,\ x_0)$ $(\varepsilon_3^\circ(t_0,\ x_0)=\varepsilon_3^*(t_0,\ x_0))$ is the value of the encounter-evasion game.

Here, when $\{t,\ x\}\notin G_2$ $(\{t,\ x\}\notin G_3)$ the mixed strategy $V_2^{(e)}(t,\ x)$ (the strategy $V_1^{(e)}(t,\ x)$) is identified with any probability measure ν $(dv\,/\,t,\ x)$ on Q (any function $v(t,\ x)\in Q$), and when $\{t,\ x\}\in G_2$, $(\{t,x\}\in G_3)$ with any probability measure $\nu^{(e)}(dv\,/\,t,\ x)$ on Q (any function $v^{(e)}(t,\ x)\in Q$) for which the condition

$$\min_{\mu}\int_P\int_Q\Big[-\frac{\partial\lambda_2(t,x)}{\partial x}\Big]'f(t,u,v)\,\mu(du)\,\nu^{(e)}\Big(\frac{dv}{t,x}\Big)=$$

$$\max_{\nu}\min_{\mu}\int_P\int_Q\Big[-\frac{\partial\lambda_2(t,x)}{\partial x}\Big]'f(t,u,v)\,\mu(du)\,\nu\,(dv)$$

$$\Big(\min_{u\in P}\Big[-\frac{\partial\lambda_3(t,x)}{\partial x}\Big]'f(t,u,v^{(e)})=\max_{v\in Q}\min_{u\in P}\Big[-\frac{\partial\lambda_3(t,x)}{\partial x}\Big]f(t,u,v)\Big)$$

is satisfied.

Above we have considered the solution of Problems 1 and 2 on ideal motions $x[t]$ which are the uniform limits of the corresponding Euler polygonal lines. However, when solving these problems we can pass to the motions realizable in practice; this can be done by turning directly to the Euler polygonal lines. The next assertion is valid.

Theorem 3.2. For any position $\{t_0, x_0\}$ and any number $\alpha > 0$ we can find a number $\delta > 0$ such that for $\Delta^{(k)} \leqslant \delta$ the inequality

$$\min_{U_i} \max_{x[\cdot]} \varphi(x_{\Delta(k)}[\cdot; t_0, x_0, U_i]) \leqslant \varepsilon_i^{\circ}(t_0, x_0) + \alpha, \quad i = 1, 2, 3$$

is satisfied for every approximate motion $x_{\Delta(k)}[t; t_0, x_0, U_i]$ [2] and the inequality

$$\max_{V_j} \min_{x[\cdot]} \varphi(x_{\Delta(k)}[\cdot; t_0, x_0, V_j]) \geqslant \varepsilon_i^{\circ}(t_0, x_0) - \alpha, \quad i = 1, 2, 3; \ j = 3, 2, 1$$

is satisfied for every motion $x_{\Delta(k)}[t; t_0, x_0, V_j]$

The given approximation in Problems 1 and 2 is stable with respect to information noise. We note that a meaningful interpretation of the solutions obtained in the class of mixed strategies can be achieved on passing to a stochastic procedure for choosing the players' controls [1, 2]. We note that in the purely linear case the problem considered in the present paper was investigated in [2].

The author thanks N.N. Krasovskii and A.I. Subbotin for valuable remarks.

REFERENCES

1. Krasovskii, N.N., Differential encounter-evasion game. I, II. Izv. Akad. Nauk SSSR, Tekhn. Kibernetika, №№ 2, 3, 1973.
2. Krasovskii, N.N. and Subbotin, A.I., Position Differential Games. Moscow, "Nauka", 1974.
3. Pontriagin, L.S., On linear differential games. I. Dokl. Akad. Nauk SSSR, Vol. 174, № 6, 1967.
4. Mishchenko, E.F., Problems of pursuit and evasion from contact in the theory of differential games. Izv. Akad. Nauk SSSR, Tekhn. Kibernetika, № 5, 1971.
5. Gusiatnikov, P.B. and Nikol'skii, M.S., On the optimality of pursuit time. Dokl. Akad. Nauk SSSR, Vol. 184, № 3, 1969.
6. Chikrii, A.A., Sufficient conditions for the completion of the differential game $z^{\cdot} = Az + f(u, v)$. Kiev, Tr. Inst. Kibernetiki Akad. Nauk UkrSSR, № 3, 1969.
7. Krasovskii, N.N. and Batukhtin, V.D., On a nonlinear differential encounter-evasion game. Dokl. Akad. Nauk SSSR, Vol. 212, № 1, 1973.
8. Krasovskii, N.N., Minimax absorption in a game of encounter. PMM Vol. 35, № 6, 1971.

Translated by N.H.C.